Journal of Statistical Physics, Vol. 52, Nos. 3/4, 1988

Chaos in Discrete Maps, Deterministic Scattering, and Nondifferentiable Functions

A. Okniński[1]

Received June 19, 1987; revision received April 1, 1988

Arguments in favor of the nondifferentiability with respect to initial data of some functions associated with deterministic discrete-time dynamical systems are presented. A correspondence between a discrete-time dynamical system and a deterministic scattering model is found and used to interpret nondifferentiability conditions. A connection with random walks is also found.

KEY WORDS: Chaotic dynamics; deterministic scattering; random walks; nondifferentiable functions.

1. INTRODUCTION

One of the characteristic properties of strange attractors is a sensitive dependence on initial conditions. Discrete-time dynamical deterministic systems, corresponding to one-dimensional maps of a given interval I,

$$a_{n+1} = f(a_n), \qquad a_n \in I \tag{1}$$

are perhaps the simplest models displaying a sensitive dependence on initial data. We shall study the dependence on initial conditions and related phenomena in the system[(1-3)]

$$a_{n+1} = 1 - \mu a_n^2, \qquad \mu \in [0, 2], \qquad a_n \in [-1, 1] \tag{2}$$

The notation $a_n = a_n(u; \mu)$, $u = a_0$, will be used to indicate the dependence of a_n on u and μ. The process (2) becomes and sensitive to initial data for $\mu > \mu_c = 1.401155$ (except from small intervals of in which a periodic

[1] Chemistry Department, Technical University (Politechnika), Noakowskiego 3, 00-664 Warsaw, Poland.

0022-4715/88/0800-0577$06.00/0

behavior is observed).[4] Intuitively, sensitivity to initial conditions of the chaotic regime may lead to pathological properties of the function $F(u; \mu, \{l_n\})$

$$F(u; \mu, \{l_n\}) = \sum_{n=0}^{\infty} l_n a_n(u; \mu), \qquad \sum_n l_n = L < \infty \tag{3a}$$

since its derivative with respect to u

$$F'(u; \mu, \{l_n\}) = \sum_{n=0}^{\infty} l_n a'_n(u; \mu) \tag{3b}$$

may be ill defined [it is assumed here that the series (3a) can be differentiated formally]. On the other hand, $F(u; \mu, \{l_n\})$ is continuous, since $|a_n| \leqslant 1$ for all n and $-L \leqslant F \leqslant L$. In the case $l_n = r^n$, $r \in [0, 1)$, $F(u; \mu, \{r^n\}) \equiv F(u; \mu, r)$ is the generating function of the sequence $\{a_n\}$

$$F(u; \mu, r) = \sum_{n=0}^{\infty} r^n a_n(u; \mu) \tag{3c}$$

The form (3a), the functional transform of the series $\sum_n a_n$, is often used in the theory of divergent series. For example, the generating function (3c) corresponds to the Abel method of summing a divergent series.

It was reported that in a discrete-time dynamical system a nondifferentiable surface may arise.[5,6] This phenomenon deserves a systematic study. Discrete-time dynamical systems of the form (1) are a good model for investigating conditions under which functional transforms corresponding to sequences $\{a_n\}$, generated according to Eq. (1), become nondifferentiable functions of the initial data (presumably in the chaotic regime).

A correspondence between the discrete-time dynamical system (2) and a very simple model of deterministic scattering is described in Section 2. Conditions leading to nondifferentiability of the associated functional transform $F(u; \mu, \{l_n\})$ are determined and expressed in terms of the Lyapunov characteristic exponents in Section 3. Computer pictures of some of the generating functions are presented in Section 4. In the last section a physical interpretation of the nondifferentiability of the generating functions (3c) is given on the basis of the deterministic scattering model. It is also shown that the same generating function (3c) is related to a random walk process.

2. A DETERMINISTIC SCATTERING MODEL

It would be useful to find a physical model which could be described in terms of the variables of a discrete-time dynamical system of the form

(1). We shall show that nondifferentiable generating functions may arise in the description of a very simple deterministic scattering model.

Let us consider a particle passing through a scattering medium. To specify the geometry of the scattering process, we use polar coordinates, taking the normal to the target as the polar axis. Since we are interested in a multiple scattering process, we consider a particle after the nth collision inside the target, incident on a scattering center. The particle direction unit vector after the nth collision Ω_n is equal to

$$\Omega_n = (\omega_n^1, \omega_n^2, \omega_n^3) = (\sin \vartheta_n \cos \varphi_n, \sin \vartheta_n \sin \varphi_n, \cos \vartheta_n) \tag{4}$$

We assume that (i) the scattering centers are infinitely heavy and that the scattering is inelastic and anisotropic; and also that the target is polarized along the polar axis, so that (ii) local coordinate systems on scattering centers are parallel.

The $\cos \vartheta_{n+1}$ of the particle after the $(n+1)$th collision is given by the spherical trigonometry formula[7]

$$\cos \vartheta_{n+1} = \cos \vartheta_n \cos \psi_n + \sin \vartheta_n \cos \chi_n \sin \psi_n \tag{5}$$

where ϑ_n is the angle with the normal, ψ_n is the angle of deviation, and χ_n is the change in the azimuth (see Fig. 1).

Let us also assume that (iii) the scattering differential cross section is such that $\psi_n = \vartheta_n$ and that (iv) $\chi_n = \chi = \text{const}$. Since the target is polarized, all scattering angles are measured in parallely transported coordinate frames on the scattering centers (Fig. 1). These two assumptions lead to deterministic scattering model

$$\omega_{n+1}^3 = \cos \chi + (1 - \cos \chi)(\omega_n^3)^2 \tag{6a}$$

$$\omega_n^3 = \cos \vartheta_n \tag{6b}$$

which can be contrasted with the Monte Carlo simulation of neutron scattering, with the angles ψ_n, χ_n treated as random numbers in $[0, \pi]$ (in the case of isotropic scattering these random numbers must be uniformly distributed in $[0, \pi]$).[7] The dynamical process of neutron scattering (6) can be considered as a caricature of a real process, yet even so simplistic a model may be useful as a basis for interpretation.

Equation (6) can be written in another form, equivalent to (2),

$$a_{n+1} = 1 - \mu a_n^2$$

$$\mu = \cos \chi(\cos \chi - 1) \tag{7a}$$

$$a_n = \omega_n^3/\cos \chi \tag{7b}$$

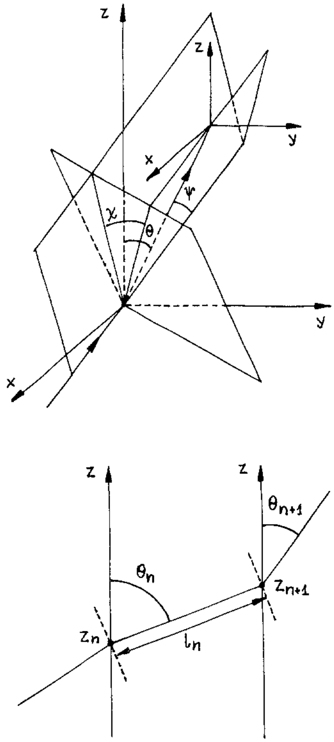

Fig. 1. Geometry and parameters of the scattering model: the angle with the z axis is ϑ, the deviation angle is ψ, the change of the azimuth due to collision is χ, the free path length after the nth collision is l_n, and the position on the z axis is z_n.

Of course, the consistency condition, $\mu \in [0, 2]$, must be fulfilled. This means that $\chi \in [\frac{1}{2}\pi, \frac{3}{2}\pi]$ must hold.

A quantity of interest is the position of the scattered particle. Let z_n be the z coordinate of the particle after the nth collision. Then

$$z_{n+1} = z_n + l_n \omega_n^3 \tag{8}$$

where l_n is the free path length after the nth collision (Fig. 1); the l_n are arbitrary apart from the condition (v) $L = \sum_{n=0}^{\infty} l_n < \infty$ (i.e., the process is inelastic), which is assumed to hold.

The z coordinate of the particle after an infinite number of collisions z_{inf} [cf. Eq. (7)],

$$z_{\text{inf}} = \cos \chi \sum_{n=0}^{\infty} l_n a_n \tag{9}$$

is proportional to the functional transform of the sequence $\{a_n\}$.

In the special case $l_n = lr^n$, where $r \in [0, 1)$ is the elasticity parameter (for $r \to 1$ the process becomes elastic), $z_{\inf}$ is proportional to the generating function of the process (2)

$$z_{\inf} = l \cos \chi \sum_{n=0}^{\infty} r^n a_n \tag{10}$$

3. CHAOTIC REGIME OF THE DISCRETE-TIME MODEL AND DIFFERENTIABILITY CONDITIONS FOR FUNCTIONAL TRANSFORMS

Differentiability conditions for the functional transforms associated with the sequence $\{a_n\}$ generated by (1) can be easily formulated in terms of the Lyapunov characteristic exponent. Let us recall the definition of the Lyapunov characteristic exponent λ,

$$\lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{df}{dx}(x_k) \right| \tag{11}$$

We get from (1)

$$x'_n = [df(x_{n-1})/dx_{n-1}]\, x'_{n-1}, \qquad x'_n = \prod_{i=0}^{n-1} df(x_i)/dx_i \tag{12}$$

where the chain rule was used and the prime indicates derivative with respect to $u = x_0$. It follows from Eqs. (11) and (12) that the asymptotics of the derivative of the functional transform (3a) is given by

$$F'(u; \mu, \{l_n\}) \sim \sum_n s_n, \qquad s_n = \operatorname{sign}(a'_n)\, l_n \exp(\lambda n) \tag{13}$$

Let us recall that the necessary convergence condition for a series $\sum_n s_n$ is of the form $s_n \to 0$ for $n \to \infty$. It thus follows that the nondifferentiability conditions for the general functional transform and for generating functions read

$$l_n \exp(\lambda n) \not\to 0 \qquad \text{for} \quad n \to \infty \tag{14a}$$

$$r \exp(\lambda) > 1, \qquad r \in [0, 1) \tag{14b}$$

respectively. The conditions (14) can be fulfilled only for $\lambda > 0$ (note that $L = \sum_n l_n$ was assumed to be finite). On the other hand, positive Lyapunov characteristic exponents mean the presence of the chaotic regime.[3]

The above arguments rely on the assumed knowledge of the Lyapunov characteristic exponent λ. On the other hand, for $\mu < 2$ the λ values were determined numerically and so are subject to some uncertainty. Furthermore, more careful analysis is needed to determine whether F' can be represented by the power series (3b).

We have calculated $F(u; \mu, r)$ for several values of r and μ for $\mu > \mu_c$ and for which the characteristic Lyapunov exponents were calculated.[8] It is important that the computer computations can be compared with exact analytic results for $\mu = 2$ since in this case the corresponding equation (2) can be solved (in this case the corresponding Lyapunov characteristic exponent is determined exactly).

The substitution

$$a_n = -\cos b_n \tag{15}$$

analogous to that of Levy and Lessman to solve a similar equation,[9] applied to Eq. (2) leads to the solution

$$a_n(u, 2) = -\cos[2^n \arccos(-u)], \qquad u \in [-1, 1] \tag{16}$$

i.e., a_n are the Tschebyschev polynomials of order 2^n.

The generating function $F(u; 2, r)$ reads

$$F(u; 2, r) = -\sum_{n=0}^{\infty} r^n \cos[2^n \arccos(-u)] \tag{17}$$

$r \in [0, 1)$, $u \in [-1, 1]$. The problem of differentiability of $F(u; 2, r)$ can be reduced to the analysis of the differentiable properties of the Weierstrass function

$$W(x; q, r) = \sum_{n=0}^{\infty} r^n \cos(q^n x), \qquad r \in [0, 1), \quad x \in (-\infty, \infty) \tag{18}$$

since

$$dF/du = -(dW/dx)(1 - u^2)^{-1/2}, \qquad x = \arccos(-u) \tag{19}$$

Weierstrass proved that for odd integer q, $q > 1 + 3\pi/2$, the function (18) was continuous and nondifferentiable. It was the first example of a function with such a property. This result was sharpened by Hardy, who demonstrated that the condition $qr > 1$ was sufficient for $W(x; q, r)$ to be nondifferentiable.[10,11] The Hardy condition can be applied directly to $F(u; 2, r)$, Eq. (17). It follows that $F(u; 2, r)$ is a continuous, nondifferentiable function of v for $r > r_c = 1/2$ and also that for $\mu = 2$ the Lyapunov characteristic exponent is equal to ln 2 [cf., Eq. (14b)].

4. COMPUTATIONAL RESULTS

The generating functions were computed for several values of μ and r from Eqs. (2) and (3c). In each case $F(u; \mu, r)$ was calculated for 2000 values of $u = a_0$ uniformly distributed in the interval $[-1, 1]$ or in its subinterval. The number of terms included in (3c) depended on r. All terms $r^n a_n$ for which the condition $r^n > \varepsilon$ was fulfilled were taken into account. The value of ε adopted in calculations varied from 10^{-4} to 10^{-9}; ε was selected as the largest number such that the form of $F(u; \mu, r)$ did not change after decrease of the value of ε.

We first computed the functions $F(u; 2, r)$ (Figs. 2–6), which, as shown in the preceding section, become nondifferentiable for $r > r_c = 1/2$ (i.e., $\lambda = \ln 2$).

Figures 2, 3, and 6 demonstrate the increasing fractal character of the curve $F(u)$. Figures 3–5 show the neighborhood of the unstable fixed point $u = 0.5$ at increasing magnification. Again, the fractal character of the curve can be seen.

Figures 7–11 show the functions $F(u; 1.684, r)$. For $\mu = 1.684$ the Lyapunov characteristic exponent determined numerically is equal 0.418,[8] and so $r_c = \ln 0.418 = 0.658$. Figures 7, 8, and 11 show again an increasing fractal character of the curve $F(u)$ for increasing r. Figures 8–10 present the neighborhood of the unstable fixed point $u = 0.5289$ at increasing magnification.

Figures 12–15 show the generating function for $\mu = 1.7664$, corresponding to the noisy three-cycle mode,[2] for which the Lyapunov

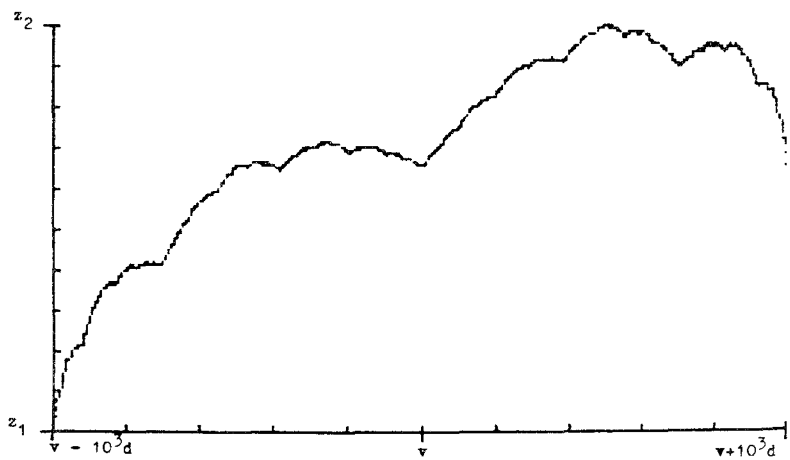

Fig. 2. The generating function $F(u; 2, r)$, $r = r_c = 0.5$ ($\lambda = \ln 2$), calculated for 2000 points uniformly distributed in the interval $[u - 10^3 d, u + 10^3 d]$, $u = 0$, $d = 10^{-3}$; $z_1 = -1.9031$, $z_2 = 1.0000$.

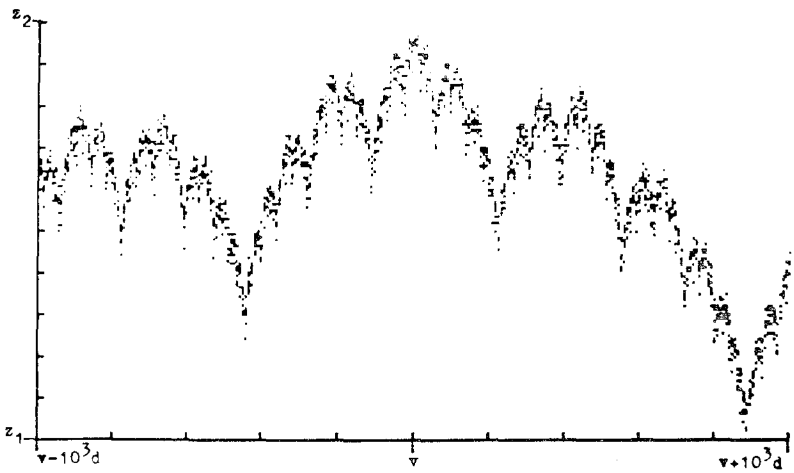

Fig. 3. Same as in Fig. 2, but with $r=0.75$, $z_1=-2.8677$, and $z_2=1.9998$.

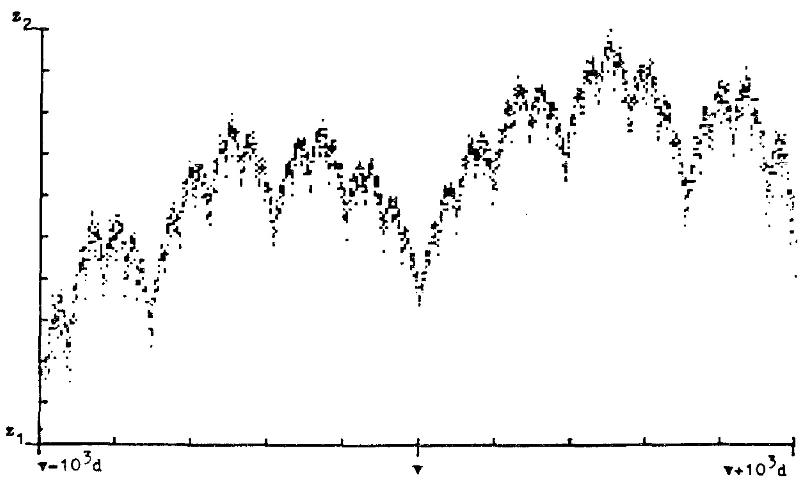

Fig. 4. Same as in Fig. 2, but with $r=0.75$, $u=0.5$, $d=10^{-6}$, $z_1=1.7587$, and $z_2=1.9998$.

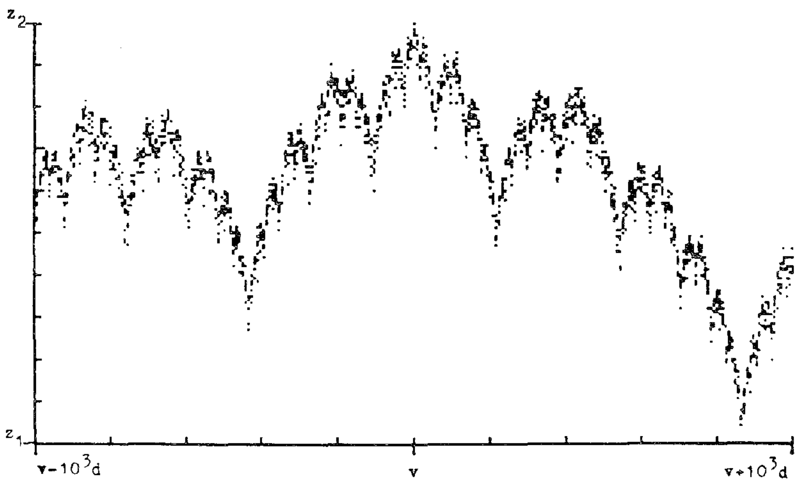

Fig. 5. Same as in Fig. 2, but with $r=0.75$, $u=0.5$, $d=10^{-9}$, $z_1=1.9861$, and $z_2=1.9999$.

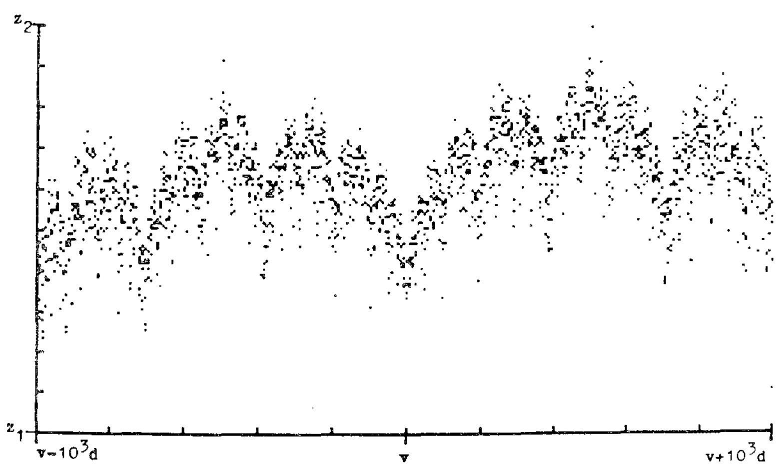

Fig. 6. Same as in Fig. 2, but with $r=0.9$, $z_1=-7.1992$, and $z_2=4.8711$.

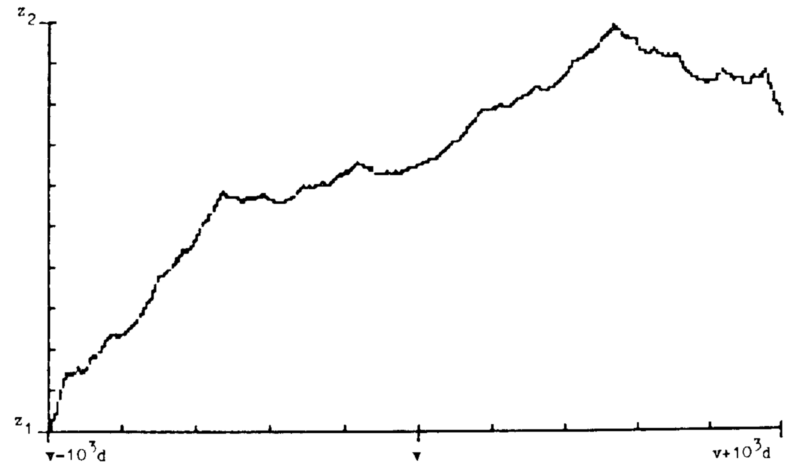

Fig. 7. The generating function $F(u;\, 1.684, r)$, $r = r_c = 0.658$ ($\lambda = 0.418$), calculated for 2000 points uniformly distributed in the interval $[u - 10^3 d, u + 10^3 d]$, $u = 0$, $d = 10^{-3}$; $z_1 = -1.0047$, $z_2 = 1.5451$.

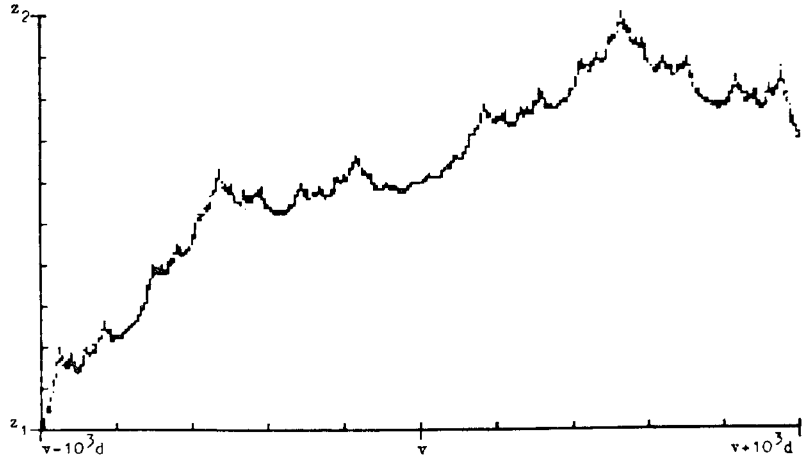

Fig. 8. Same as in Fig. 7, but with $r = 0.75$, $z_1 = -0.7561$, and $z_2 = 2.1012$.

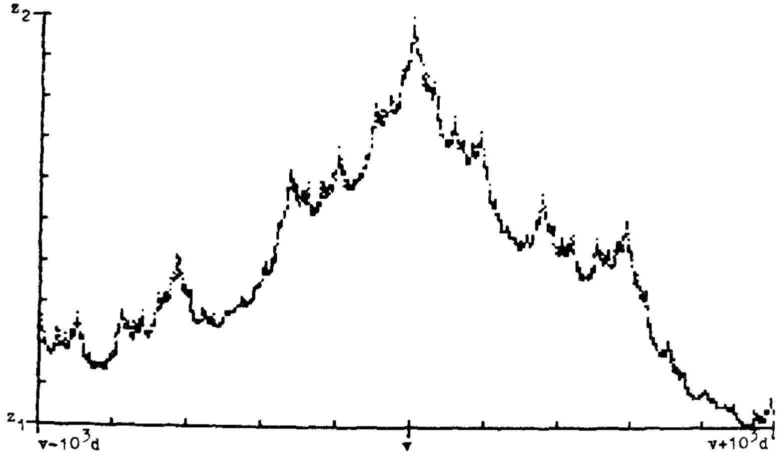

Fig. 9. Same as in Fig. 7, but with $r=0.75$, $u=0.5289093$, $d=10^{-6}$, $z_1=2.0702$, and $z_2=2.1155$.

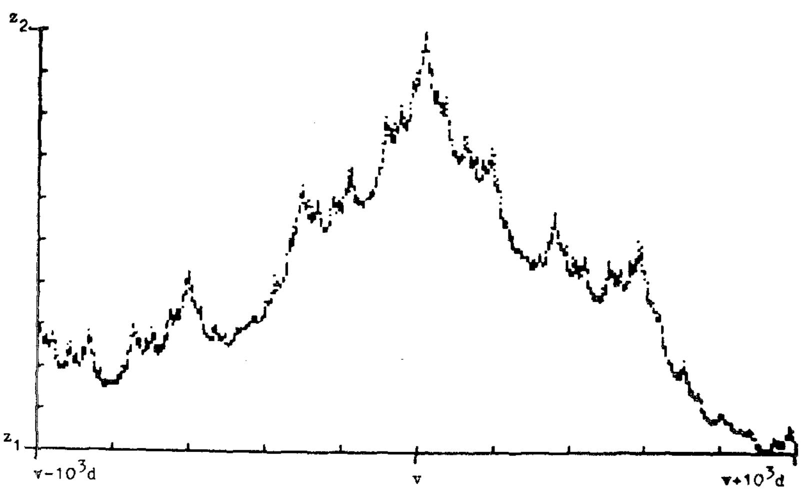

Fig. 10. Same as in Fig. 7, but with $r=0.75$, $u=0.5289093$, $d=10^{-9}$, $z_1=2.1142$, and $z_2=2.1156$.

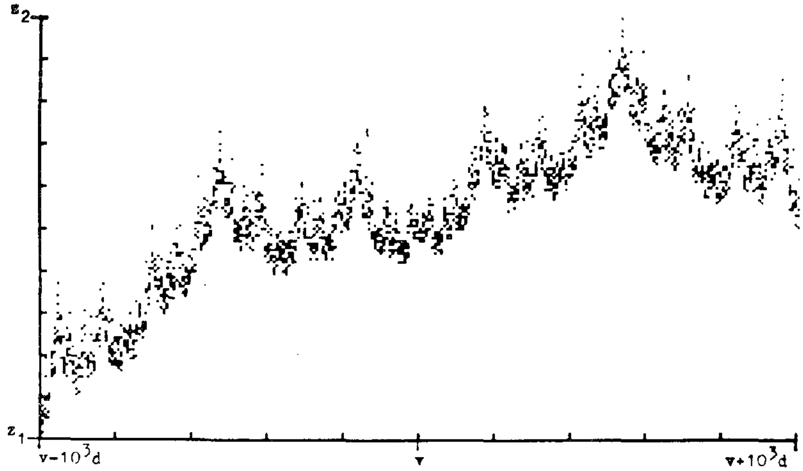

Fig. 11. Same as in Fig. 7, but with $r=0.9$, $z_1=0.9278$, and $z_2=4.7700$.

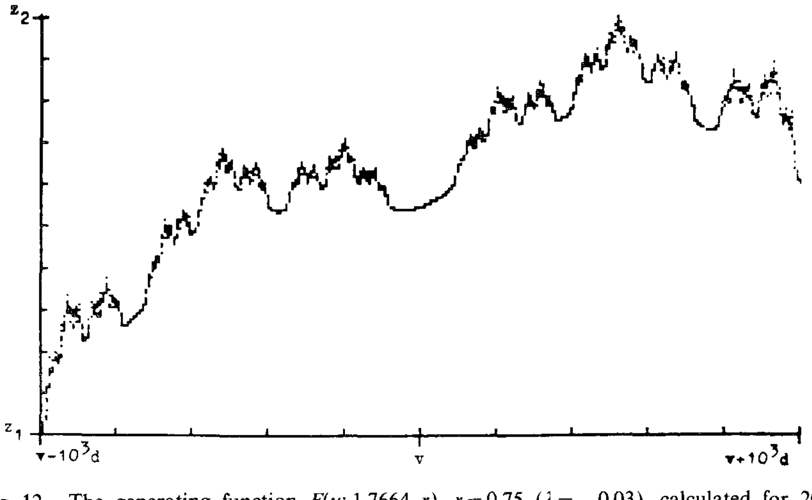

Fig. 12. The generating function $F(u;\, 1.7664, r)$, $r=0.75$ ($\lambda=-0.03$), calculated for 2000 points uniformly distributed in the interval $[u-10^3d, u+10^3d]$, $u=0$, $d=10^{-3}$; $z_1=-1.2855$, $z_2=2.0503$.

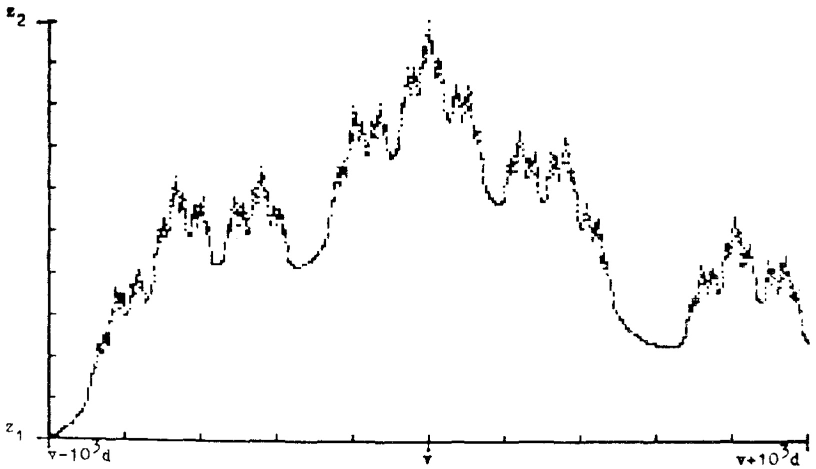

Fig. 13. Same as in Fig. 12, but with $u = 0.5208333$, $d = 10^{-6}$, $z_1 = 2.0058$, and $z_2 = 2.0830$.

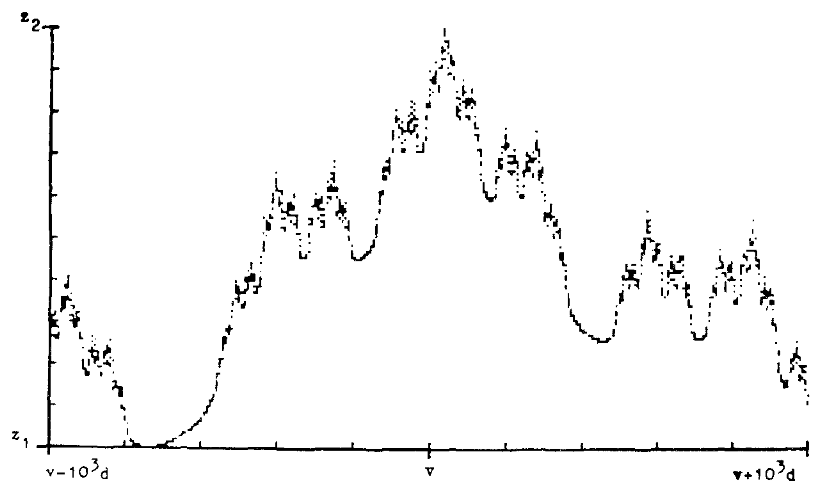

Fig. 14. Same as in Fig. 12, but with $u = 0.5208333$, $d = 10^{-9}$, $z_1 = 2.0808$, and $z_2 = 2.0833$.

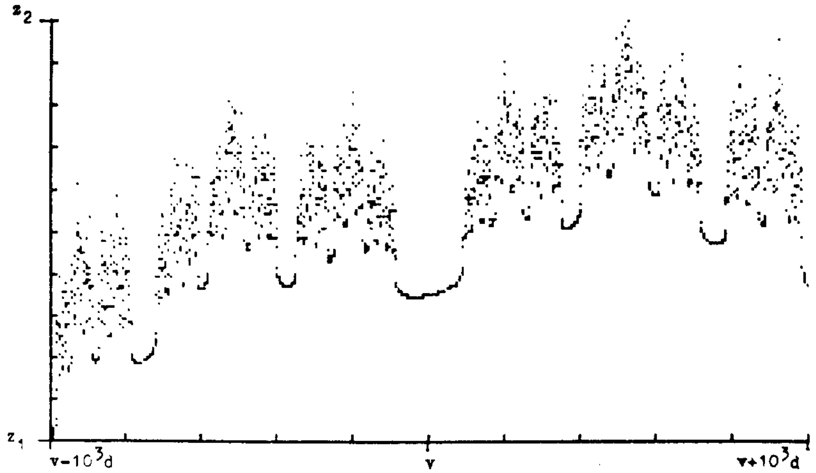

Fig. 15. Same as in Fig. 12, but with $r=0.9$, $z_1=-0.9227$, and $z_2=4.4869$.

characteristic exponent is negative, $\lambda=-0.03$.[8] Figures 12 and 15 demonstrate the dependence on r, while Figs. 12–14 show the neighborhood of the unstable fixed point $u=0.5208$ at increasing magnification. The curves are piecewise smooth and have complicated structure in between. It follows, however, that they are differentiable, since for $\mu=1.7664$ the Lyapunov characteristic exponent is negative (Section 3).

5. DISCUSSION

Implications of the analytic and computational results for the deterministic scattering model described in Section 2 are discussed in Section 5.1. Connections with random walk processes are pointed out in Section 5.2. Both models are compared in Section 5.3.

5.1. Deterministic Scattering Model

The quantity z_{inf}/l can be written in the form

$$z_{\text{inf}}/l=\cos(\chi)\,F[\cos(\vartheta_0)/\cos(\chi);\,\mu,\,r] \tag{20}$$

[cf. Eqs. (2), (3c), (7b), and (10). It is a continuous and also a differentiable or nondifferentiable function of ϑ_0 (where ϑ_0 is the initial angle between the velocity vector of the particle and the z axis), depending on the choice of the parameters χ, r.

More precisely, the continuous function $z_{\inf}(\vartheta_0)$ should be nondifferentiable for μ corresponding to the chaotic regime of the process (2), where $\mu = \cos\chi(\cos\chi - 1)$ and $r > r_c$, with $r_c = \exp(\lambda)$ and λ the Lyapunov characteristic exponent [cf. (14b)]. The angle χ corresponds to the change in the azimuth (Fig. 1), while r controls the elasticity of the scattering process; the free path length after the nth collision is equal to $l_n = lr^n$, so that the process is elastic for $r = 1$.

The theoretical results of Section 3 can be applied directly to a more general scattering model (9), provided that the parameters $\{l_n\}$ as well as the Lyapunov characteristic exponent λ are known. The nondifferentiability condition in this case is given by (14a).

We close our discussion with some more general remarks. Neutron scattering is a good example of Brownian motion and it can be simulated by Monte Carlo techniques. We have substituted the process of stochastic scattering by a deterministic model. It would be interesting to see whether the model of Section 2 leaves some properties of the Brownian motion unchanged.

Let us recall that the path of a particle undergoing Brownian motion is continuous and nondifferentiable.[12] Let an initial distribution of Brownian particles be spherically symmetric. Then, after a finite period of diffusion, a new surface is formed. This surface is nondifferentiable since every particle path is nondifferentiable.

Let us now consider the deterministic scattering model of Section 2. The three-dimensional generalization of Eqs. (6) and (8) reads [cf. Eq. (4)]

$$(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n) + l_n(\omega_n^1, \omega_n^2, \omega_n^3) \tag{21}$$

$$\omega_{n+1}^3 = \cos\chi + (1 - \cos\chi)(\omega_n^3)^2$$

$$\omega_{n+1}^{1,2} = \{\omega_n^{1,2}\omega_n^3(1 - \omega_{n+1}^3) \mp \sin\chi\, \omega_n^{2,1}[1 - (\omega_n^3)^2]\}/[1 - (\omega_n^3)^2] \tag{22}$$

Let us assume that an initially spherically symmetric cloud of neutrons with a uniform distribution of initial values of ϑ_0, φ_0 is scattered. Then an initial spherically surface of the neutron cloud (x_0, y_0, z_0) transforms after an infinite number of collisions into a two-dimensional surface $(x_{\inf}, y_{\inf}, z_{\inf})$, nondifferentiable with respect to the initial parameter ϑ_0 (and hence x_0, y_0, z_0) in the chaotic regime of the process (2), controlled by the parameter χ and for an appropriate choice of another control parameter r. Thus, in the chaotic regime of the deterministic scattering model, Eqs. (21), (6), and (22), some properties of the Brownian motion of indeterministically scattered particles are preserved.

5.2. Random Walks

It is interesting that the Weierstrass function (18) is closely related to the random walk problems.[13] Namely, let us consider a one-dimensional walk on a periodic infinite lattice. The following possible lengths of jumps with their appropriate probabilities are postulated (we use notation of ref. 13):

$$\begin{aligned} &\pm 1 \text{ with probability } Cl_0, \qquad \pm b \text{ with probability } Cl_1,\ldots \\ &\pm b^j \text{ with probability } Cl_j, \qquad l_j > 0, \quad j = 0, 1, 2,\ldots \end{aligned} \tag{23}$$

where C is the normalization constant, $C = 1/(2L)$, $L = \sum_j l_j < \infty$. Then the probability of a given displacement being equal to l is

$$p(l) = (1/2L) \sum_{j=0}^{\infty} l_j(\delta_{l,b^j} + \delta_{l,-b^j}) \tag{24}$$

The structure function for the random walk $p(k)$, i.e., the Fourier transform of $p(l)$, is equal to

$$p(k) = \sum_l p(l) \exp(ilk) = (1/L) \sum_{j=0}^{\infty} l_j \cos(b^j k) \tag{25a}$$

It follows from (3a), (16), and (25a) that for $b = 2$ the structure function is proportional to the functional transform of the sequence $\{a_n(u; 2)\}$, $u = -\cos k$, generated according to Eq. (2)

$$p(k) = -(1/L) \sum_{j=0}^{\infty} l_j a_j(-\cos k; 2) = -(1/L) F(-\cos k; 2, \{l_j\}) \tag{26a}$$

In the special case considered in ref. 13, $l_j = N^{-j}$, we get [cf. Eq. (18)]

$$p(k) = \frac{N-1}{N} \sum_{j=0}^{\infty} N^{-j} \cos(b^j k) = \frac{N-1}{N} W(k; b, N^{-1}) \tag{25b}$$

It follows from (17) and (25b) that for $b = 2$ the structure function of the Weierstrass random walk $p(k; 2, r) \equiv p(k)$ is proportional to the generating function $F(u; \mu, r)$

$$p(k; 2, r) = -(1-r) F(-\cos k; 2, r), \qquad N^{-1} = r, \quad L^{-1} = 1 - r \tag{26b}$$

It can be shown that the trajectory of the Weierstrass random walk, performed on a periodic one-dimensional lattice, traces out self-similar clusters; this follows from the fractal structure of the Weierstrass function.[13] Let us recall here that also $F(u; \mu, r)$ for $\mu > \mu_c$, $r > r_c$, are fractal.

Let us consider now a general generating function (3c). It follows from Eqs. (26b), (6b), and (7b) that for $\mu < 2$ we should consider the form $\{-\text{const} \cdot F[\cos(k)/\cos(\chi); \mu, r]\}$ as the definition of a structure function $p(k; \mu, r)$ of some random walk process [see also Eqs. (28)]:

$$p(k; \mu, r) \underset{\text{df}}{=} -\text{const} \cdot F[\cos(k)/\cos(\chi); \mu, r] \tag{27a}$$

Obviously, the Fourier transform of $p(k; \mu, r)$,

$$p(l; \mu, r) = \sum_k p(k; \mu, r) \exp(-ilk) \tag{27b}$$

i.e., the probability of a given displacement being equal to l, must not contain negative terms.

We have performed test computations of Fourier transforms of $\{-F[\cos(k)/\cos(\chi); \mu, r]\}$. The Fourier transforms of the generating functions are real and positive within numerical accuracy for all $\mu \in [0, 2]$. Furthermore, for $\mu \cong 2$, $p(l; \mu, r)$ is very similar in form to $p(l; 2, r)$, i.e., to the Fourier transform of $-(1-r) F(-\cos k; 2, r)$, though there are small nonzero coefficients in the Fourier spectrum for some l not equal to 2^j, $j = 0, 1, 2,\ldots$. For μ significantly smaller than 2 the contributions for $l = 2^j$ still dominate, but contributions from $l \neq 2^j$ are not negligible.

We can thus associate with discrete maps of form (2) a random walk, defining the corresponding structure function as follows:

$$\begin{aligned} p(k; \mu, r) &= [1/C(\mu, r)]\{-F[\cos(k)/\cos(\chi); \mu, r]\} \\ &= [1/C(\mu, r)] \sum_{n=0}^{\infty} c_n(\mu, r) \cos(nk) \end{aligned} \tag{28a}$$

$$C(\mu, r) = \sum_n c_n(\mu, r) \tag{28b}$$

The probability of a displacement being equal to l is thus given by

$$p(l; \mu, r) = (1/2C) \sum_{n=0}^{\infty} c_n(\mu, r)(\delta_{l,n} + \delta_{l,-n}), \qquad c_n \geqslant 0 \tag{29}$$

and for $\mu \cong 2$ approximates well the random walk process (23), (24).

For $\mu < 2$, apart from possible jumps of lengths $\pm 2^j$ (of high probability), jumps of other lengths are possible, including the possibility that the particle does not change its position $[c_0(\mu, r) > 0]$.

5.3. Closing Remarks

We compare briefly the two models described in Sections 5.1 and 5.2. Notice that the two dimensionless quantities z_{inf}/l and $p(k; \mu, r)$ [cf. Eqs. (20) and (28), respectively] depend on the generating function (3c) in the same manner. It is interesting that while z_{inf}/l is a deterministic quantity, the Fourier transform of $p(k; \mu, r)$, $p(l; \mu, r)$, has a probabilistic interpretation.

Let us also note here that in the deterministic models (1) random-type behavior was observed. Numerical results for the map

$$a_{n+1} = a_n - \mu \sin(2\pi a_n) \tag{30}$$

show that for $\mu > \mu_c = 0.73264$ the process can leave the interval $[-1/2, 1/2]$ and a random walk on the real axis is performed.[14]

REFERENCES

1. R. M. May, *Nature* **262**:459 (1976).
2. E. Ott, *Rev. Mod. Phys.* **53**:655 (1981).
3. J.-P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**:617 (1985).
4. M. J. Feigenbaum, *J. Stat. Phys.* **19**:25 (1978); **21**:669 (1979); *Phys. Lett. A* **74**:375 (1979); *Commun. Math. Phys.* **77**:65 (1980).
5. J. L. Kaplan, J. Mallet-Paret, and J. A. Yorke, University of Maryland, Preprint (1982).
6. J. D. Farmer, E. Ott, and J. A. Yorke, *Physica* **7D**:105 (1983).
7. H. Kahn, *Nucleonics* **6**:27 (1950).
8. J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, *Phys. Rep.* **92**:45 (1982).
9. H. Levy and F. Lessman, *Finite Difference Equations* (MacMillan, New York, 1961).
10. G. H. Hardy, *Trans. Am. Math. Soc.* **17**:301 (1916).
11. A. Zygmund, *Duke Math. J.* **12**:47 (1945).
12. R. E. A. C. Paley and N. Wiener, *Fourier Trnsforms in the Complex Domain* (American Mathematical Society, Providence, Rhode Island, 1934).
13. E. W. Montroll and M. F. Shlesinger, in *Nonequilibrium Phenomena. II. From Stochastics to Hydrodynamics*, J. L. Lebowitz and E. W. Montroll, Eds. (North-Holland, New York, 1984).
14. T. Geisel and J Nierwetberg, in *Lecture Notes in Physics*, Vol. 179 (Springer, New York, 1983), p. 93.